#### Converse coding theorems for quantum source and noisy channel

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#### Abstract

The weak converse coding theorems have been proved for the quantum source and channel. The results give the lower bound for capacity of source and the upper bound for capacity of channel. The monotonicity of mutual quantum information have also been proved.

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#### 1 Introduction

There is currently much interest in quantum information theory, the theory of compressing, transmitting and storing of messages, represented as a some quantum states [1-2,4-12]. The concepts of information can be properly formulated only in the context of physical theory. Therefore the physics of information is interestingly from many standpoints, not only because of its obvious practical and engineering importance. When quantum effects become important, for example at the level of single electrons and photons, the existing classical information theory becomes fundamentally inadequate.

In general, quantum information theory contains two distinct types of problems. The the first type describes transmission of classical information through a channel (the channel can be noisy or noiseless). In other words bits encoded as some quantum states and only this states or its tensor products are transmitted. In the second type of problems an

arbitrary superposition of this states or entanglement states are transmitted. In the first case the problems can be solved by methods of classical information theory, at least in principle. But in the second case new physical representations are needed.

Quantum information theory has some application in the theory of nonideal quantum computers [2]. This hypothetical computational machines work by laws of quantum mechanics and can efficiently solve some problems that are believed to be intractable on any classical computer. However, when we considerate physically realizable quantum computers, we must consider errors due to the computers-environment coupling. In other words decoherence and noise penetrate in to the computer. For this reason at present the formulation of quantum error-correcting codes is intensively being studied [7-10]. Unlike error-correcting codes the subject of quantum information theory is not a study of realizable in practice codes, but investigation of general limits for compression and transmission rates. It is correct to say that basic research in information theory is directed towards the understanding of general restrictions of information compressing and transmitting without taking into account their practical applications. The general problems of information theory are coding problems for source (noiseless channel) and noisy channel [6,7,12,13]. Let us briefly explain what each of them means. A quantum source is determined by a set of states, each state can be used with probability  $p_i$ , i = 1, ..., N.

$$\rho = \sum_{i} p_{i} |\psi_{i}\rangle\langle\psi_{i}|, \quad \sum_{i} p_{i} = 1.$$
 (1)

We only assume that  $\langle \psi_i | \psi_i \rangle = 1$  and the states may be nonorthogonal. There are many quantum sources for fixed  $\rho$ . We consider the general quantum evolutions operators  $\hat{S}_1, \hat{S}_2$  [4,14] as coding and decoding. This operators must be linear, completely positive and trace-preserving. In this paper we consider only unit-preserving quantum evolutions operators.

$$\hat{S}\rho = \sum_{\mu} A^{\dagger}_{\mu} \rho A_{\mu}, \quad \sum_{\mu} A_{\mu} A^{\dagger}_{\mu} = \hat{1}, \quad \sum_{\mu} A^{\dagger}_{\mu} A_{\mu} = \hat{1},$$
 (2)

where  $\{A_{\mu}\}$  is a some set of operators. In particular  $\hat{S}_1$ ,  $\hat{S}_2$  may be linear combinations of unitary operations, measurements, partial traces and others.

The problem of coding is generated by a practical problems, which are connected with

compressing of a message or a data transferring in channel without noise. We define the coding as follows

$$|\psi_i\rangle\langle\psi_i|\mapsto \hat{S}_1|\psi_i\rangle\langle\psi_i|=\pi_i, \quad \rho\mapsto\rho_1=\sum_{i=1}p_i\pi_i.$$
 (3)

And in decoding we have

$$\pi_i \mapsto \hat{S}_2 \pi_i = w_i, \quad \rho_1 \mapsto \rho_2 = \sum_{i=1} p_i w_i. \tag{4}$$

Where dim  $\rho = \dim \rho_2 \ge \dim \rho_1$ . We must get  $\hat{S}_1, \hat{S}_2$  with minimal dim  $\rho_1$  for fixed  $\rho$  and fidelity  $F_e$  [3] close to 1 (the degree of closeness is also fixed).

$$F_e = \sum_{i,j} p_i p_j \langle \psi_i | \hat{S}_1 \hat{S}_2(|\psi_i\rangle \langle \psi_j|) |\psi_j\rangle.$$
 (5)

There are several definitions of fidelity in literature [3,5,12]. This quantity must characterize the degree of closeness between two density matrices and equal to 1 if and only if this density matrices are identical. Author of [4] uses for fidelity another expression without referring to a concrete representation of  $\rho$ .

$$F_e = \langle \psi^R | \hat{S}_1 \hat{S}_2(|\psi^R\rangle \langle \psi^R) | \psi^R \rangle. \tag{6}$$

Where  $\psi^R$  is a purification of  $\rho$ 

$$|\psi^R\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle \otimes |\phi_i^R\rangle, \quad \langle \phi_j^R |\phi_i^R\rangle = \delta_{ij},$$
 (7)

$$tr_R |\psi^R\rangle\langle\psi^R| = \rho,$$
 (8)

here  $\{|\phi_i^R\rangle\}$  is some orthonormal set. The definition is independent to the concrete choice of this set [1].

We pint out that definition of fidelity in the form [6,13]

$$\bar{F} = \sum_{i} p_i \langle \psi_i | w_i | \psi_i \rangle \tag{9}$$

is nonadequate in general case. As example we consider the fully decoherent mapping

$$A_{\mu} = |\mu\rangle\langle\mu|, \quad \langle\mu|\acute{\mu}\rangle = \delta_{\mu\acute{\mu}}. \tag{10}$$

Now for the density matrix  $\sigma$ 

$$\sigma = \sum_{\mu} c_{\mu} |\mu\rangle\langle\mu| \tag{11}$$

we have  $\bar{F} = 1$  but  $F_e = \sum_{\mu} |c_{\mu}|^2 \le 1$ . Indeed in general case we have [4]

$$\bar{F} \ge F_e.$$
 (12)

In other words we may say that  $F_e$  taking into account the deformation of phases for  $\psi_i$ . We define the rate of coding for source as [3]

$$R = \log_2 \dim \rho_1. \tag{13}$$

This quantity characterizes the degree of compressing for  $\rho$ .

Now we describe the coding problem for a quantum noisy channel, defined by mapping  $\hat{S}$ . We see that the meaning of a source coding is an extracting of a redundancy. In noisy channel coding we introduce redundancy, as a result we have faithful transporting of density matrix  $\rho$ . Now the  $\hat{S}_1$ ,  $\hat{S}_2$  are also coding and decoding procedures.

$$\rho \mapsto \hat{S}_1 \rho \mapsto \hat{S} \hat{S}_1 \rho \mapsto \hat{S}_2 \hat{S} \hat{S}_1 \rho = \rho_2. \tag{14}$$

Here dim  $\rho = \dim \rho_2 \leq \dim \hat{S}_1 \rho$ . We must find  $\hat{S}_1$ ,  $\hat{S}_2$  for fixed  $\rho$  with fidelity  $F_e$  between  $\rho$  and  $\rho_2$  close to 1.

In section 3 we define the problems statements more precisely and introduce block coding. In this work we get converse theorems for quantum source and noisy channel. Quantum source were considered in [6,13]. In [6] the author proved the direct coding theorem for source. In [13] authors considered a general nonunitary decoding, but used  $\bar{F}$  for fidelity. Authors of [6,13] worked with memoryless source. A quantum channels considered in [6,11].

In section 2 the concept of quantum information is reviewed [5]. Using general properties of relative entropy, the theorem about monotonicity of the mutual quantum information is proved. In section 3 our results are discussed.

### 2 Relative entropy and quantum mutual information

Almost all quantities in quantum information theory are formulated in terms of von Newmans entropy

$$S(\rho) = -tr\rho \log_2 \rho. \tag{15}$$

This quantity characterize the degree of 'unorder' of  $\rho$  and invariant with respect to unitary transformation of  $\rho$ . The main quantity in classical information theory is the mutual information between two ensembles of random variables X, Y.

$$I(X,Y) = H(Y) - H(Y/X).$$
 (16)

This is the decrease of entropy of X due to the knowledge of Y, and conversely with interchanging X and Y. Here H(Y) and H(Y/X) are Shannon entropy and conditional entropy [3]. In particular X, Y may be the input and the output of a noisy channel. In 1989 M.Ohya [11] made an attempt to introduce mutual information in quantum theory. But the mutual information in quantum theory must take into account a specific character of quantum information transmission. More reasonable definition of quantum mutual information was introduced by B.Schumacher and M.P. Nielsen [5] (in this work authors call this quantity coherent information). These authors connected mutual information with the deformation degree of initial density matrix purification (8,9)

$$I(\rho; \hat{S}) = S(\hat{S}\rho) - S(\hat{1}^R \otimes \hat{S}(|\psi^R\rangle\langle\psi^R|)), \tag{17}$$

$$\hat{1}^R \otimes \hat{S}(|\psi^R\rangle\langle\psi^R|)) = \sum_{i,j} \sqrt{p_i p_j} |\phi_i^R\rangle\langle\phi_j^R| \otimes \hat{S}(|\psi_i\rangle\langle\psi_j|).$$
 (18)

Quantum mutual information is the decrease of the entropy after acting of  $\hat{S}$  due to the possible distortion of the entanglement state. This value is not symmetric with respect to interchanging of input and output, and can be positive, negative or zero (in classical case mutual information is a nonnegative value).

Quantum relative entropy between two density matrices  $\rho_1$ ,  $\rho_2$  is defined as follows

$$S(\rho_1||\rho_2) = tr(\rho_1 \log \rho_1 - \rho_1 \log \rho_2). \tag{19}$$

This quantity was introduced by Umegaki [15] and characterizes the degree of 'closeness' of density matrices  $\rho_1$ ,  $\rho_2$ . The properties of quantum relative information were reviewed by M.Ohya [11]. Here only one basic property is mentioned.

$$S(\rho_1||\rho_2) \ge S(\hat{S}\rho_1||\hat{S}\rho_2). \tag{20}$$

This inequality was proved by Lindblad [16] and valid not only for unit-preserving maps, but also in general case. As a consequence of Lindblad inequality we can prove the monotonicity of mutual information [5].

$$I(\rho; \hat{S}_1) \ge I(\rho; \hat{S}_2 \hat{S}_1).$$
 (21)

Indeed we have

$$S(\hat{1}^R \otimes \hat{S}(|\psi^R\rangle\langle\psi^R|)||\hat{1}^R \otimes \hat{S}(\rho^R \otimes \rho))$$

$$= -S(\hat{1}^R \otimes \hat{S}(|\psi^R\rangle\langle\psi^R|)) + S(\rho^R) + S(\hat{S}\rho)). \tag{22}$$

Here

$$\rho^{R} = \sum_{i,j} \sqrt{p_{i}p_{j}} |\phi_{i}^{R}\rangle \langle \phi_{j}^{R} | \langle \psi_{i} | \psi_{j} \rangle.$$
 (23)

Now from Lindblad inequality we have

$$S(\hat{1}^R \otimes \hat{S}(|\psi^R\rangle\langle\psi^R|)||\hat{1}^R \otimes \hat{S}(\rho^R \otimes \rho))$$

$$\geq S(\hat{1}^R \otimes \hat{S}_1 \hat{S}_2(|\psi^R\rangle\langle\psi^R|)||\hat{1}^R \otimes \hat{S}_1 \hat{S}_2(\rho^R \otimes \rho)). \tag{24}$$

From Lindblad inequality and (2) we have that if  $\dim \rho = \dim \hat{S}\rho$  and unit-preserving  $\hat{S}$ .

$$S(\hat{S}\rho) \ge S(\rho). \tag{25}$$

This inequality is an analog of L.Boltzmans H-theorems.

Further  $S(\hat{1}^R \otimes \hat{S}(|\psi^R\rangle\langle\psi^R|))$  will be written as  $S(\rho \mapsto \hat{S}\rho)$ .

In (3) B.Schumacher proved quantum Fano inequality.

$$S(\rho \mapsto \hat{S}\rho) \le (1 - F_e) \log_2(d^2 - 1) + h(F_e),$$
 (26)

$$h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x), \tag{27}$$

where  $d = \dim \hat{S}\rho$ . In qualitative level the meaning of this theorem is as follows: If we connect  $1 - F_e$  with probability of error, then  $h(1 - F_e)$  is an information for decision. And if we have error, then  $(1 - F_e) \log_2(d^2 - 1)$  is the upper bound of information for determination of this error.

## 3 The converse coding theorems

For generality reasons we formulate our theorems in terms of blocks with length n. In other words instead of  $\rho$ 

$$\rho^{(n)} = \rho \otimes \dots \otimes \rho \tag{28}$$

is used. The direct theorems of information theory frequently has only asymptotic character and valid only for  $n \gg 1$  [3]. Now all transformations  $\hat{S}_1^{(n)}$ ,  $\hat{S}_2^{(n)}$ ,  $\hat{S}_2^{(n)}$ ,  $\hat{S}_2^{(n)}$  acts on  $\rho^{(n)}$  and if  $\hat{S}^{(n)} = \hat{S} \otimes ... \otimes \hat{S}$  then we have memoryless channel (source). We working with any n and with channel (source) with any memory.

In quantum information theory we have von Newman entropy as an analog of corresponding quantity H(X), introduced by Shannon [3]. For the source coding the following theorem is proved:

If  $R \leq S(\rho)$  then  $F_e$  between  $\rho^{(n)}$  and  $\rho_2^{(n)}$  is not close to 1 for any n,  $\hat{S}_1^{(n)}$ ,  $\hat{S}_2^{(n)}$ .

In other words there exist some  $\delta$  and  $F_e \leq \delta < 1$ . In section 4 the problem of existence for a strong converse theorem in a general case is discussed.

With the help of (15,19) we get

$$nR \ge S(\rho_1^{(n)}) \ge I(\rho^{(n)}; \hat{S}_1^{(n)}).$$
 (29)

From the (23) we obtain the following formula

$$nR > I(\rho^{(n)}; \hat{S}_2^{(n)} \hat{S}_1^{(n)}) = S(\rho_2^{(n)}) - S(\rho^{(n)} \mapsto \rho_2^{(n)}). \tag{30}$$

With the help of H-theorem we come to

$$nR \ge nS(\rho) - S(\rho^{(n)} \mapsto \rho_2^{(n)}). \tag{31}$$

Now if  $R = S(\rho) - \delta$ , with  $\delta \ge 0$  from quantum Fano inequality we have

$$\delta \le ((1 - F_e)\log_2(d^{2n} - 1) + h(F_e))/n. \tag{32}$$

We see that for any n,  $F_e$  is not close to 1. If  $n \to \infty$  then

$$\delta \le 2(1 - F_e)\log_2 d. \tag{33}$$

Capacity  $C_s$  of a quantum source is defined as follows.

If  $R \geq C_s$  then there exist some  $\hat{S}_2^{(n)}$ ,  $\hat{S}_1^{(n)}$  with  $F_e$  is close to 1. If  $R \leq C_s$  then for any  $\hat{S}_2^{(n)}$ ,  $\hat{S}_1^{(n)}$   $F_e$  is not close to 1.

In [6] B.Schumacher proved existence of  $\hat{S}_{2}^{(n)}$ ,  $\hat{S}_{1}^{(n)}$  with  $\bar{F}$  close to 1 and  $\log_2 \dim \rho_1 = nS(\rho)$  for large n. Our inequality show, that for sufficiently general coding and decoding schemes and fidelity  $F_e$  we have  $C_s \geq S(\rho)$ .

Now we pass to the converse theorem for noisy quantum channel.

In a channel coding we introduce redundancy in the step  $\rho \mapsto \rho^{(n)}$  (in source coding problem we directly work with  $\rho^{(n)}$ ). After this  $\hat{S}_1^{(n)}$  is a unitary transformation. The concrete form of this transformation depend from concrete coding schemes [8-10]. Decoding transformation  $\hat{S}_1^{(n)}$  may be nonunitary, because it contain a determination of unknown quantum state. In other words in channel coding  $\hat{S}_1^{(n)}$  is a unitary transformation,  $\hat{S}_2^{(n)}$  is a general but unit-preserving. We introduce the quantity  $\tilde{C}$  as

$$\tilde{C} = \max_{\{p_i\}} I(\rho^{(n)}; \hat{S}^{(n)})/n. \tag{34}$$

Here  $\hat{S}^{(n)}$  is a quantum channel. We define the rate of noisy channel coding as

$$R_c = \max_{\{p_i\}} S(\rho) \tag{35}$$

If  $\langle \psi_i | \psi_j \rangle = \delta_{ij}$  we have a classical-like input ensemble of random variables, and  $S(\rho^n)$  is maximized by the uniform distribution of  $p_i$  and the famous classical expression for  $R_c$  [3] is obtained

$$R_c = \log_2 d. (36)$$

For channel coding we will prove the following theorem.

If  $R_c \geq \tilde{C}$ , then  $F_e$  (between  $\rho^{(n)}$  and  $\rho_2^{(n)}$ ) is not close to 1 for any n,  $\hat{S}_1^{(n)}$ ,  $\hat{S}_2^{(n)}$ .

In other, words exist some  $\delta$ , that  $F_e \leq \delta < 1$ .

Derivation of this theorem is similar to source coding theorem. We have from (23)

$$I(\rho^{(n)}; \hat{S}^{(n)}) \ge I(\rho^{(n)}; \hat{S}_2^{(n)} \hat{S}_1^{(n)} \hat{S}_1^{(n)}).$$
 (37)

Now with  $\tilde{C}$  we get to the following chain of inequalities for any set  $\{p_i\}$ 

$$n\tilde{C} \geq I(\rho^{(n)}; \hat{S}^{(n)})$$

$$\geq I(\rho^{(n)}; \hat{S}_{2}^{(n)} \hat{S}^{(n)} \hat{S}_{1}^{(n)})$$

$$\geq S(\rho_{2}^{(n)}) - S(\rho^{(n)} \mapsto \rho_{2}^{(n)}), \tag{38}$$

$$n\tilde{C} \geq S(\rho^{(n)}) - S(\rho^{(n)} \mapsto \rho_2^{(n)})$$

$$= nS(\rho) - S(\rho^{(n)} \mapsto \rho_2^{(n)}). \tag{39}$$

Now we may work with the set  $\{p_i\}$ , which maximized input entropy (36)

$$n\tilde{C} \geq nR_c - S(\dots \mapsto \dots)$$

$$S(\dots \mapsto \dots) \geq (R_c - \tilde{C})n$$

$$(R_c - \tilde{C})n \leq ((1 - F_e)\log_2(d^{2n} - 1) + h(F_e))/n. \tag{40}$$

Capacity  $C_c$  of a quantum channel is defined as following.

If  $R \leq C_c$  then for channel  $\hat{S}^{(n)}$  exists some coding and decoding procedures  $\hat{S}_2^{(n)}$ ,  $\hat{S}_1^{(n)}$  with  $F_e$  between input and output is close to 1.

If  $R \geq C_s$  then for any  $\hat{S}_2^{(n)}$ ,  $\hat{S}_1^{(n)}$   $F_e$  is not close to 1.

We see, that indeed  $\tilde{C} \geq C_c$ . The question about direct coding theorem for some restricted class of channels in quantum information theory is still open.

### 4 Conclusion

Recently in literature the question about capacity of a quantum channel was discussed. Quantum Fano theorem was proved in [3]. But Fano theorem is not a general converse theorem. Fano-like theorem may be written for several independent quantities [11]. As it has been mentioned, the authors of [12] considered the capacity of quantum source for general decoding mapping, but with nonadequate fidelity.

Our results suggest that in sufficiently general case the capacity of a quantum source indeed equal  $S(\rho)$ . The upper bound for the capacity of quantum noisy channel has been found. This quantity is less than other candidates for a quantum channel capacities role [12].

In general case the proposition of the weak converse theorem cannot be strengthened [3]. Strong converse theorem has the following structure.

If  $R \geq \tilde{C}$  is true for quantum channel  $(R \leq S(\rho))$  for source) then for  $n \gg 1$  and any  $\hat{S}_1$ ,  $\hat{S}_2$   $F_e$  is close to 0.

In classical theory [3] strong converse theorems can be proved only for simple memoryless (may be in some effective sense, see [3]) channels or sources. A proof of such theorems in quantum theory for general coding and decoding schemes is still an open problem. In the [7] author in qualitative manner considered the direct coding problem for memoryless channel. He gave some arguments that  $\tilde{C} \leq C_c$ , but did not consider the entanglement fidelity.

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